

ON SPECTRAL ASYMPTOTICS OF QUASI-EXACTLY SOLVABLE QUARTIC AND YABLONSKII-VOROB'EV POLYNOMIALS

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ABSTRACT. Motivated by the earlier results of [15] and [22], we study theoretically and numerically the asymptotics and the monodromy of the quasi-exactly solvable part of the spectrum of the quasi-exactly solvable quartic introduced by C. M. Bender and S. Boettcher [1]. In particular, we formulate a conjecture on the coincidence of the asymptotic shape of the configuration of the branching points of the latter quartic with the asymptotic shape of zeros of the Yablonskii-Vorob'ev polynomials recently described in [9]–[10] and present its (conjectural) alternative description, see Conjecture 4.

1. INTRODUCTION

A quasi-exactly solvable quantum-mechanical quartic oscillator was introduced in [1] and (in its restricted form) is a Schrödinger-type eigenvalue problem of the form

$$L_J(y) = y'' - (x^4/4 - ax^2/2 - Jx)y = \lambda y \quad (1.1)$$

with the boundary conditions $y(te^{\pm\pi I/3}) \rightarrow 0$ as $t \rightarrow +\infty$, where $a \in \mathbb{C}$ and J are parameters of the problem.¹ With these boundary conditions, real a and J , (1.1) is not Hermitian but PT -symmetric, see [20], [15] and [6]. When $J = n + 1$ is a positive integer, then $L_{n+1}(y)$ maps the space of quasi-polynomials $\{pe^h : \deg p \leq n\}$ to itself where p is a polynomial of degree at most n and $h = -x^3/6 + ax/2$. The restriction of L_{n+1} to the latter space is a finite-dimensional linear operator whose spectrum and eigenfunctions can be found explicitly. This part of the

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¹This paper written by a mathematician and a physicist contains three types of results: rigorous mathematical statements, statements proven of the physics level of rigor, i.e., under certain assumptions of convergence and, finally, conjectures obtained after substantial numerical experimentation. It best fits under the category of experimental mathematics and illustrates the sagacious saying of the late Vladimir Arnold:

”Mathematics is a branch of physics in which experiments are cheap.”

spectrum and eigenfunctions of (1.1) is usually referred to as *quasi-exactly solvable*.

With a slight abuse of notation, one can easily show that polynomial factors in the quasi-exactly solutions of (1.1) are exactly polynomial solutions of the degenerate Heun equation

$$y'' - (x^2 - a)y' + (\alpha x - \lambda)y = 0, \quad (1.2)$$

where $a \in \mathbb{C}$ has the same meaning as above and (α, λ) are spectral variables. Obviously, if equation (1.2) has a polynomial solution of degree n , then $\alpha = n$. Additionally, to get a polynomial solution of (1.2) of degree n , the remaining spectral variable λ should be chosen as an eigenvalue of the operator

$$T_n(y) = y'' - (x^2 - a)y' + nxy$$

acting on the linear space of polynomials of degree at most n . In the standard monomial basis $\{1, x, x^2, \dots, x^n\}$ the operator T_n acts by the 4-diagonal $(n+1) \times (n+1)$ square matrix $M_n^{(a)}$ given by:

$$M_n^{(a)} := \begin{pmatrix} 0 & a & 2 & 0 & 0 & \cdots & 0 \\ n & 0 & 2a & 6 & 0 & \cdots & 0 \\ 0 & n-1 & 0 & 3a & 12 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 & 0 & (n-1)a & n(n-1) \\ 0 & 0 & \cdots & 0 & 2 & 0 & na \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (1.3)$$

We will call the bivariate polynomial $Sp_n(a, \lambda) := \det(M_n^{(a)} - \lambda I)$ the *n-th spectral polynomial of (1.2)*. The total degree of $Sp_n(a, \lambda)$ equals $n+1$ which is also its degree with respect to the variable λ . (The degree of the variable a in $Sp_n(a, \lambda)$ equals $\lfloor \frac{n+1}{2} \rfloor$.) Observe that if a is a real number then $Sp_n(a, \lambda)$ is a real polynomial in λ and therefore the spectrum is symmetric with respect to the real axis.

Our main goal is to study the behavior of the spectrum of $M_n^{(a_n)}$ when $n \rightarrow \infty$ for two regimes of a , namely, when $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$ or when $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = a \neq 0$. In other words, we consider solutions of $Sp_n(a_n, \lambda) = 0$ with respect to λ in the above two situations. Observe that if $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}}$ does not exist one can not expect any reasonable asymptotic behavior of the spectrum.

Remark. In case of a non-degenerate Heun equation detailed study of similar asymptotics was carried out in [19], [22] by the present authors jointly with K. Takemura.

Our mathematically rigorous result is as follows.

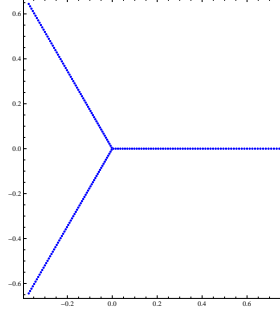


FIGURE 1. Distributions of the eigenvalues of $M_n^{(0)}$ scaled by $n^{4/3}$ for $n=200$.

Theorem 1. (i) If $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$, the maximal absolute value $r_n(a_n)$ of the eigenvalues of $M_n^{(a_n)}$, i.e., the maximal absolute value among the roots of $Sp_n(a_n, \lambda)$, grows as $\frac{3}{4}n^{4/3}$.

(ii) If $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$, the sequence $\{\mu_n^{(a_n)}\}$ of root-counting measures for the sequence $\{Sp_n(a_n, \beta n^{4/3})\}$, i.e., for the sequence of spectra of $\{M_n^{(a_n)}\}$ rescaled so that each eigenvalue of $M_n^{(a_n)}$ is divided by $n^{4/3}$, weakly converges to the measure ν_0 supported on the union of three straight intervals connecting the origin with three cubic roots of unity multiplied by $\frac{3}{4}$, see Figure 1.

More information about ν_0 can be found in § 2.

Our main result on a physics level of rigor is as follows. Consider the family

$$\mathcal{C}^2 - (x^2 - a)\mathcal{C} + (x - \beta) = 0 \quad (1.4)$$

of quadratic equations in \mathcal{C} depending on parameters a and β .

Proposition 2. *If $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = a \neq 0$, the sequence $\{\mu_n^{(a_n)}\}$ of root-counting measures for $\{Sp_n(a_n, \beta n^{4/3})\}$ weakly converges to a compactly supported probability measure ν_a . The support of ν_a consists of all values of β , for which there exists a compactly supported in the x -plane probability measure κ whose Cauchy transform $\mathcal{C}_\kappa(x)$ satisfies (1.4) almost everywhere in the x -plane.*

Recall that for a (complex-valued) measure μ compactly supported in \mathbb{C} , its logarithmic potential is defined as

$$\mathbf{u}_\mu(x) := \int_{\mathbb{C}} \ln |x - \xi| d\mu(\xi)$$

and its Cauchy transform is defined as

$$\mathcal{C}_\mu(x) := \int_{\mathbb{C}} \frac{d\mu(\xi)}{x - \xi} = \frac{\partial \mathbf{u}_\mu(x)}{\partial x}.$$

The existence of a signed (not necessarily positive) measure μ whose Cauchy transform $\mathcal{C}_\mu(x)$ satisfies (1.4) almost everywhere in \mathbb{C} is closely related to the properties of the family

$$\Psi_{a,\beta} = -((x^2 - a)^2 - 4(x - \beta))dx^2 \quad (1.5)$$

of quadratic differentials depending on a and β . More information about this connection can be found in § 3.

Remark 1. For basic information on quadratic differentials and its horizontal trajectories consult [21]. Proposition 2 reminds of the main result of [4], where we considered the so-called homogenized spectral problem. In this case all terms of the Heun differential operator in question are important for the root asymptotics of sequences of eigenpolynomials. The validity of Proposition 2 was also verified numerically.

Finally our main conjecture supported by extensive numerical experiments is as follows. For a given positive integer n , denote by Σ_n the set of all branching points of the projection of the algebraic curve $\Gamma_n(a) : \{Sp_n(a, \lambda) = 0\}$ to the a -axis. In other words, Σ_n is the set of all values of the complex parameter a for which $Sp_n(a, \lambda)$ has a multiple root in λ . Equivalently, Σ_n is the zero locus of the univariate discriminant polynomial $Dscr(Sp_n(a, \lambda))$ which is the resultant of $Sp_n(a, \lambda)$ with $\frac{\partial Sp_n(a, \lambda)}{\partial \lambda}$. One can show that the degree of the latter polynomial equals $\binom{n+1}{2}$.

Yablonskii-Vorob'ev polynomials are defined as follows, see [24] – [25]. Set $YV_0 = 1$, $YV_1 = t$. For $n \geq 1$ set

$$YV_{n+1} = \frac{t \cdot YV_n^2 - 4(YV_n \cdot YV_n'' - YV_n')^2}{YV_{n-1}}.$$

Although the latter expression apriori determines a rational function, YV_n is in fact a polynomial of degree $\binom{n+1}{2}$, see e.g. [23]. The importance of Yablonskii-Vorob'ev polynomials is explained by the fact that all rational solutions of the second Painlevé equation

$$u_{tt} = tu + 2u^3 + \alpha, \alpha \in \mathbb{C},$$

are presented in the form

$$u(t) = u(t; n) = \frac{d}{dt} \left\{ \ln \left[\frac{YV_{n-1}(t)}{YV_n(t)} \right] \right\}, \quad u(t, 0) = 0, \quad u(t; -n) := -u(t; n).$$

Denote by $\{\mathcal{Z}_n\}$ the zero loci of $\{YV_n\}$.

Remark 2. One can show that the maximal absolute value of points in Σ_n grows as $\frac{3}{\sqrt[3]{4}}n^{2/3}$ while the maximal absolute value of points in \mathcal{Z}_n grows as $\left(\frac{9}{2}\right)^{\frac{2}{3}}n^{2/3}$.

Conjecture 1. (i) *After division by $\frac{3}{\sqrt[3]{4}}n^{2/3}$ and $(\frac{9}{2})^{\frac{2}{3}}n^{2/3}$ respectively, the sequence $\{\Sigma_n\}$ tends to the sequence $\{\mathcal{Z}_n\}$. In particular, the limiting triangular domain \mathfrak{F} covered by $\{\Sigma_n\}$ and $\{\mathcal{Z}_n\}$ when $n \rightarrow \infty$ is the same.*

(ii) *The boundary of \mathfrak{F} consists of all values of a , for which the support of measure ν_a consists of a single arc with a singular point on it.*

For more details on Conjecture 1 see § 4 and for its illustration see Fig. 2.

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2. CASE WHEN $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$. MATHEMATICS RIGOR.

We start with the spectral asymptotics in the basic case $a_n = 0$, $n = 1, 2, 3, \dots$

Proposition 3. (i) The spectrum of $M_n^{(0)}$, i.e., the zero locus of $Sp_n(0, \lambda)$, is invariant under the rotation by $2\pi/3$ around the origin, i.e., it has a \mathbb{Z}_3 -symmetry.

(ii) The sequence $Sp_n(0, \lambda)$ splits into the following three subsequences:

- (1) if $n + 1 = 3k$, the polynomial $Sp_n(0, \lambda)$ contains only the third powers of λ and denoting $\xi = \lambda^3$, we have $Sp_n(0, \lambda) = q_k^{(0)}(\xi)$;
- (2) if $n + 1 = 3k + 1$, the polynomial $Sp_n(0, \lambda)$ can be factorized as $Sp_n(0, \lambda) = \lambda q_k^{(1)}(\xi)$;
- (3) if $n + 1 = 3k + 2$, the polynomial $Sp_n(0, \lambda)$ can be factorized as $Sp_n(0, \lambda) = \lambda^2 q_k^{(2)}(\xi)$.

(iii) All three polynomials $q_k^{(0)}(\xi)$, $q_k^{(1)}(\xi)$, $q_k^{(2)}(\xi)$ have the same degree k ; all their roots are negative and simple which implies that the spectrum is located on the union of three rays through the origin as illustrated in Figure 1.

To prove Proposition 3 following [14], we need to introduce a double indexed polynomial sequence containing our original sequence $\{Sp_n(0, \lambda)\}$. The most natural way to do this is to consider the principal minors of $M_n(0) - \lambda I$ given by:

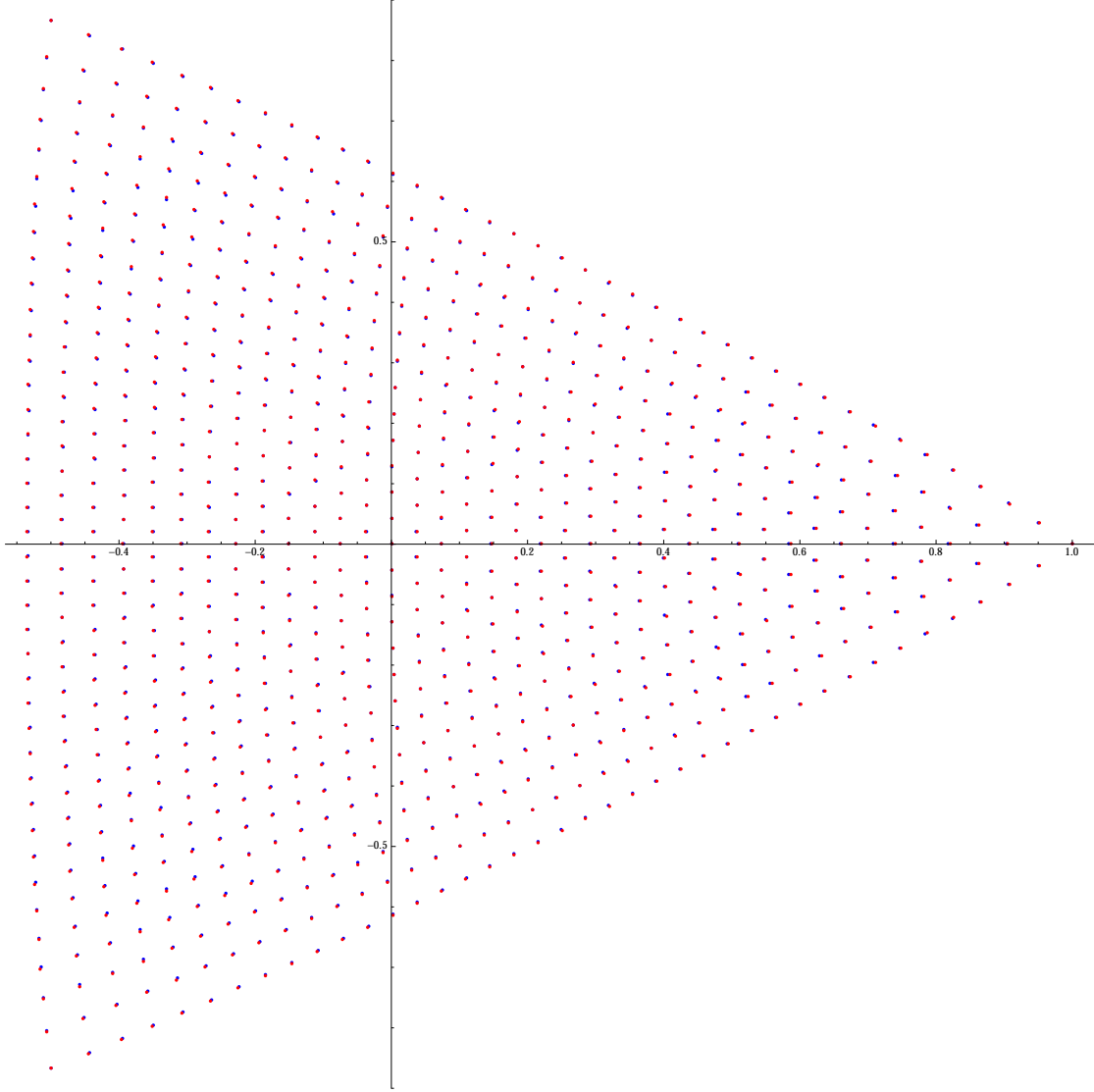


FIGURE 2. The set Σ_{40} of the branching points for $Sp_{40}(a, \lambda)$ together with the zero locus \mathcal{Z}_{40} after scaling. (Observe the surprising closeness of the scaled Σ_{40} and \mathcal{Z}_{40} . Their distinction is hardly visible by a naked eye!)

$$M_n^{(0)} - \lambda I := \begin{pmatrix} -\lambda & 0 & 2 & 0 & 0 & \cdots & 0 \\ n & -\lambda & 0 & 6 & 0 & \cdots & 0 \\ 0 & n-1 & -\lambda & 0 & 12 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 & -\lambda & 0 & n(n-1) \\ 0 & 0 & \cdots & 0 & 2 & -\lambda & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & -\lambda \end{pmatrix}. \quad (2.1)$$

Namely, denote by $\Delta_n^{(k)}(\lambda)$ the k -th principal minor of (2.1). Obviously, $Sp_n(0, \lambda) = \Delta_n^{(n+1)}(\lambda) = \det(M_n(0) - \lambda I)$. Since the latter matrix is 4-diagonal with one subdiagonal and two superdiagonals, then, by a general result of [17], its principal minors satisfy a linear recurrence relation of order at most 3 (i.e., a 4-term relation). Simple explicit calculation gives

$$\Delta_n^{(k)}(\lambda) = -\lambda \Delta_n^{(k-1)}(\lambda) + (n-k+2)(n-k+3)(k-1)(k-2) \Delta_n^{(k-3)}(\lambda), \quad (2.2)$$

where k runs from 1 to $n+1$, with the standard boundary conditions

$$\Delta_n^{(-2)}(\lambda) = \Delta_n^{(-1)}(\lambda) = 0, \quad \Delta_n^{(0)}(\lambda) = 1.$$

Recurrence (2.2) has variable coefficients depending both on k and n .

Another form of (2.2) which looks more promising is as follows. To simplify the signs, let us consider the characteristic polynomials of the principal minors of $M_n^{(0)} + \lambda I$ (instead of $-\lambda I$ as above). With a slight abuse of notation, the recurrence relation looks as:

$$\Delta_n^{(k)}(\lambda) = \lambda \Delta_n^{(k-1)}(\lambda) + (n-k+2)(n-k+3)(k-1)(k-2) \Delta_n^{(k-3)}(\lambda). \quad (2.3)$$

Proof of Proposition 3. Items (i) and (ii) follow immediately from the next statement.

Lemma 4. Denote $\Delta_n^{3l}(\lambda) = P_l(\xi)$, $\Delta_n^{3l+1}(\lambda) = \lambda Q_l(\xi)$, $\Delta_n^{3l+2}(\lambda) = \lambda^2 R_l(\xi)$, where $\xi = \lambda^3$ and P_l, Q_l, R_l are monic polynomials of degree l .

Then in terms of P_l, Q_l, R_l recurrence (2.3) looks as:

$$\begin{cases} P_l(\xi) = \xi R_{l-1}(\xi) + (n-3l+2)(n-3l+3)(3l-2)(3l-1)P_{l-1}(\xi) \\ Q_l(\xi) = P_l(\xi) + (n-3l+1)(n-3l+2)(3l-1)3lQ_{l-1}(\xi) \\ R_l(\xi) = Q_l(\xi) + (n-3l)(n-3l+1)3l(3l+1)R_{l-1}(\xi) \end{cases} \quad (2.4)$$

with the initial conditions $P_0(\xi) = Q_0(\xi) = R_0(\xi) = 1$. Here l runs from 1 to $[n/3]$.

Proof. Simple algebra. \square

To prove item (iii) of Proposition 3, we will use notation of Lemma 4. Hence P_k, Q_k, R_k correspond to $q_k^{(0)}, q_k^{(1)}$ and $q_k^{(2)}$ respectively. Our first step is to prove that each polynomial in the recurrences has positive coefficients and real roots. Moreover the roots of the polynomials appearing in the same recurrence are interlacing. Note that, in our case, positive coefficients imply that all roots are negative.

We will use induction on l . As the base of induction we check case $l = 2$ since case $l = 1$ is trivial. Elementary calculations give $\xi R_1 =$

$\xi^2 + (84 - 8n + 20n^2)\xi$ and $P_1 = \xi - 2n + 2n^2$, which proves that the roots of each one of them are negative and are interlacing for all integer $n \geq 3$. Hence, since $P_2 = \xi R_1 + 20(n - 4)(n - 3)P_1$, then using Lemma 1.10 of [13] or conducting elementary calculations, we can conclude that P_2 has real roots. Using similar elementary calculations we obtain similar results for the pairs P_2, Q_1 and Q_2, R_1 .

Assume now that our hypothesis holds for a given positive integer l . Note that, in recurrence for l , degrees of the polynomials differ by one. This indicates that the largest root belongs to the polynomial with larger degree. Furthermore by Corollary 1.30 of [13], we can derive the following results:

$$\xi R_{l-1} \leftarrow P_l \leftarrow P_{l-1} \quad (2.5)$$

$$P_l \leftarrow Q_l \leftarrow Q_{l-1} \quad (2.6)$$

$$Q_l \leftarrow R_l \leftarrow R_{l-1}. \quad (2.7)$$

Here the arrow " \leftarrow " indicates that the corresponding pair of polynomials have interlacing roots with the largest root belonging to the polynomial at which the arrow points.

(a) Consider the recurrence

$$P_{l+1} = \xi R_l + (n - 3l - 1)(n - 3l)(3l + 1)(3l + 2)P_l.$$

We can rewrite ξR_l as $\xi Q_l + (n - 3l)(n - 3l + 1)3l(3l + 1)\xi R_{l-1}$. Observe that $\xi R_{l-1} \leftarrow P_l$ by induction hypothesis and Corollary 1.30 in [13]. Using (2.2), we can conclude that $\xi Q_l \leftarrow P_l$ since all roots of P_l are negative. Hence $\xi R_l \leftarrow P_l$, by Lemma 1.31 in [13]. Therefore P_{l+1} has real roots, by Lemma 1.10 in [13]. Furthermore, since P_l and R_l have positive coefficients, so does P_{l+1} , hence it has negative roots.

(b) Consider the recurrence

$$Q_{l+1} = P_{l+1} + (n - 3l - 2)(n - 3l - 1)(3l + 2)(3l + 3)Q_l.$$

One can rewrite P_{l+1} as in part (a). By (2.2), $P_l \leftarrow Q_l$ and by (2.3), $\xi R_l \leftarrow Q_l$ since R_l and Q_l only have negative roots. Proof is concluded by the same argument as in part (a).

(c) Consider the recurrence

$$R_{l+1} = Q_{l+1} + (n - 3l - 3)(n - 3l - 2)(3l + 3)(3l + 4)R_l.$$

One can rewrite Q_{l+1} as in part (b). By (2.3), $Q_l \leftarrow R_l$. $P_{l+1} \leftarrow R_l$ by part (a), Corollary 1.30 in [13] and the fact that both P_{l+1} and R_l have only negative roots. Proof is concluded by the same argument as in parts (a) and (b).

Second step is to prove that all roots of the considered polynomials are simple. Assume that P_l has a double root p . Then by Corollary 1.30 in [13], p should also be a root of ξR_{l-1} and P_{l-1} . Hence when we conduct our induction on l and assume that ξR_{l-1} and P_{l-1} have only

simple roots, we get a contradiction since P_l is a linear combination of those polynomials. Note that, for $l = 2$, we only have simple zeros. The same argument applies to Q_l and R_l . \square

Proof of Theorem 1 in case $a_n = 0$. In order to apply the approach of [14], we need the recurrence coefficients to stabilize when $\frac{k}{n} \rightarrow \tau$, for any fixed $\tau \in [0, 1]$. This stabilization would hold if instead of (2.2) one considers

$$\tilde{\Delta}_n^{(k)}(\beta) = -\beta \tilde{\Delta}_n^{(k-1)}(\beta) + \frac{(n-k+2)(n-k+3)(k-1)(k-2)}{n^4} \tilde{\Delta}_n^{(k-3)}(\beta), \quad (2.8)$$

where $\lambda = n^{4/3}\beta$. Namely, when $\frac{k}{n} \rightarrow \tau$, the latter relation tends to following relation with constant coefficients:

$$\tilde{\Delta}_\tau^{(k)}(\beta) = -\beta \tilde{\Delta}_\tau^{(k-1)}(\beta) + (1-\tau)^2 \tau^2 \tilde{\Delta}_\tau^{(k-3)}(\beta). \quad (2.9)$$

Observe that (2.8) is the recurrence for the principal minors of the matrix $\frac{1}{n^{4/3}} M_n^{(0)} - \beta I$ given by:

$$\frac{1}{n^{4/3}} M_n^{(0)} - \beta I := \begin{pmatrix} -\beta & 0 & \frac{2}{n^{4/3}} & 0 & 0 & \cdots & 0 \\ \frac{n}{n^{4/3}} & -\beta & 0 & \frac{6}{n^{4/3}} & 0 & \cdots & 0 \\ 0 & \frac{n-1}{n^{4/3}} & -\beta & 0 & \frac{12}{n^{4/3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{3}{n^{4/3}} & -\beta & 0 & \frac{n(n-1)}{n^{4/3}} \\ 0 & 0 & \cdots & 0 & \frac{2}{n^{4/3}} & -\beta & 0 \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{n^{4/3}} & -\beta \end{pmatrix}. \quad (2.10)$$

In other words, (2.8) is satisfied by the characteristic polynomials of the principal minors of $\frac{1}{n^{4/3}} M_n^{(0)}$. Following [14], we conclude that the Cauchy transform (considered in the complement $\mathbb{C} \setminus \Omega$ where Ω is an appropriate compact subset) of the asymptotic distribution of the sequence $\{Sp_n(0, \beta)\} = \{Sp_n(0, \frac{\lambda}{n^{4/3}})\} = \{\tilde{\Delta}_n^{(n+1)}(\beta)\}$ is obtained by averaging the Cauchy transforms of the asymptotic distributions of (2.9) over $\tau \in [0, 1]$.

In fact, in the case under consideration even the density of the former distribution can be obtained by averaging the densities of the latter family of distributions.

Recurrence (2.9) is similar to the one considered in the last section of [3] and has very nice asymptotic distribution of its roots, see Figure 3. Observe that for any $\tau \in [0, 1]$, the initial conditions for (2.9) are taken as

$$\tilde{\Delta}_\tau^{(-2)}(\beta) = \tilde{\Delta}_\tau^{(-1)}(\beta) = 0, \tilde{\Delta}_\tau^{(0)}(\beta) = 1.$$

Since (2.9) has constant coefficients, the support of the asymptotic root-counting measure of its solution is given by the well-known result

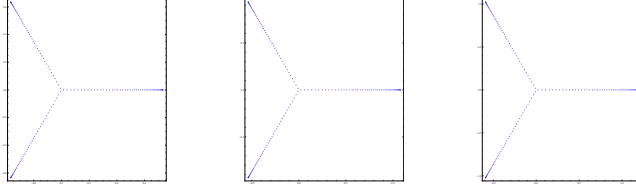


FIGURE 3. Root distributions for $n = 150$ of the solution to the recurrence relation (2.9) for $\tau = 1/4$, $\tau = 1/2$ and $\tau = 3/4$ resp.

of Beraha-Cahane-Weiss [2]. Namely, it coincides with the set of all $\beta \in \mathbb{C}$ such that the characteristic equation

$$\Psi^3 + \beta\Psi^2 - (1 - \tau)^2\tau^2 = 0 \quad (2.11)$$

has two solutions with respect to Ψ which have the same modulus and, additionally, this modulus is the largest among all three solutions of (2.11). From considerations of [3] one can easily derive that, for any fixed $\tau \in [0, 1]$, this support is the union of three intervals starting at the origin and ending at the branching points of (2.11) which are given by the equation:

$$\beta^3 = \frac{27}{4}(1 - \tau)^2\tau^2. \quad (2.12)$$

These branching points are located as shown in Figure 3. Observe that if for $\tau \in [0, 1]$, $\beta^+(\tau)$ denotes the branching point lying on the positive half-axis, then it attains its maximum at $\tau = 1/2$ and this maximum equals $\frac{3}{4}$.

As we mentioned before, in the case under consideration all supports of (2.9) lie on three fixed rays through the origin. Therefore, the density of the asymptotic distribution of the sequence

$$\{Sp_n(0, \beta)\} = \{Sp_n(0, \frac{\lambda}{n^{4/3}})\} = \{\tilde{\Delta}_n^{(n+1)}(\beta)\}$$

is obtained by averaging the densities of (2.9) over $\tau \in [0, 1]$, see [12] and [14]. Therefore, the support of the asymptotic root distribution for $\{Sp_n(0, \beta)\}$ is the union of three intervals of length $\frac{3}{4}$. \square

Proof of Theorem 1 when $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$. We have to show that Proposition 3 holds asymptotically for any sequence $\{a_n\}$ satisfying the condition $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$. The principal minors of $M_n(a_n) - \lambda I$ given by:

$$M_n^{(a_n)} - \lambda I := \begin{pmatrix} -\lambda & a_n & 2 & 0 & 0 & \cdots & 0 \\ n & -\lambda & 2a_n & 6 & 0 & \cdots & 0 \\ 0 & n-1 & -\lambda & 3a_n & 12 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 & -\lambda & (n-1)a_n & n(n-1) \\ 0 & 0 & \cdots & 0 & 2 & -\lambda & na_n \\ 0 & 0 & \cdots & 0 & 0 & 1 & -\lambda \end{pmatrix}, \quad (2.13)$$

satisfy the recurrence

$$\begin{aligned} \Delta_n^{(k)}(a_n, \lambda) &= -\lambda \Delta_n^{(k-1)}(a_n, \lambda) + (k-1)(n-k+2)a_n \Delta_n^{(k-2)}(a_n, \lambda) \\ &\quad + (n-k+2)(n-k+3)(k-1)(k-2) \Delta_n^{(k-3)}(a_n, \lambda), \end{aligned} \quad (2.14)$$

where k runs from 1 to $n+1$, with the boundary conditions

$$\Delta_n^{(-2)}(a_n, \lambda) = \Delta_n^{(-1)}(a_n, \lambda) = 0, \quad \Delta_n^{(0)}(a_n, \lambda) = 1.$$

To obtain a converging sequence of root-counting measures one has to consider

$$\begin{aligned} \tilde{\Delta}_n^{(k)}(a_n, \beta) &= -\beta \tilde{\Delta}_n^{(k-1)}(a_n, \beta) - \frac{(k-1)(n-k+2)a_n}{n^{8/3}} \tilde{\Delta}_n^{(k-2)} \\ &\quad + \frac{(n-k+2)(n-k+3)(k-1)(k-2)}{n^4} \tilde{\Delta}_n^{(k-3)}(\beta), \end{aligned} \quad (2.15)$$

which is the recurrence for the principal minors of $\frac{1}{n^{4/3}} M_n^{(a_n)} - \beta I$ given by:

$$\begin{pmatrix} -\beta & \frac{a_n}{n^{4/3}} & \frac{2}{n^{4/3}} & 0 & 0 & \cdots & 0 \\ \frac{n}{n^{4/3}} & -\beta & \frac{2a_n}{n^{4/3}} & \frac{6}{n^{4/3}} & 0 & \cdots & 0 \\ 0 & \frac{n-1}{n^{4/3}} & -\beta & \frac{3a_n}{n^{4/3}} & \frac{12}{n^{4/3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{3}{n^{4/3}} & -\beta & \frac{(n-1)a_n}{n^{4/3}} & \frac{n(n-1)}{n^{4/3}} \\ 0 & 0 & \cdots & 0 & \frac{2}{n^{4/3}} & -\beta & \frac{na_n}{n^{4/3}} \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{n^{4/3}} & -\beta \end{pmatrix}. \quad (2.16)$$

When $\frac{k}{n} \rightarrow \tau$, the family (2.15) of recurrence relations converges to the earlier family (2.9) valid in case $a_n = 0$ due to the fact that $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$. Therefore the measure obtained by averaging of the family of root-counting measures for recurrence relations with constant coefficient is exactly the same as in case $a_n = 0$, i.e., it coincides with ν_0 . Additionally observe that by a general result of [14], the support of the asymptotic root-counting measure of the sequence $\{\tilde{\Delta}_n^{(k)}(a_n, \beta)\}$ can only be smaller than that of ν_0 and their Cauchy transforms coincide outside the support of ν_0 . Since the support of ν_0

is the union of three straight intervals through the origin the resulting asymptotic root-counting measure for the sequence $\{Sp_n(a_n, \beta)\}$ in case $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = 0$ coincides with ν_0 . \square

Remark 3. Since the support of the limiting measure ν_0 consists of three segments through the origin it is possible in principle to write down integral formulas for the density and the Cauchy transform of ν_0 similar to those presented in [14], [12] and [19]. In particular, in the complement to the support of ν_0 its Cauchy transform is given by

$$\mathcal{C}_{\nu_0}(\beta) = \int_0^1 \frac{\tilde{\Psi}'_{\beta} d\tau}{\tilde{\Psi}},$$

where $\tilde{\Psi}$ is the unique solution of (2.11) satisfying $\lim_{\beta \rightarrow \infty} \frac{\tilde{\Psi}}{\beta} = -1$ and $\tilde{\Psi}'_{\beta}$ is its partial derivative with respect to β . However it seems difficult to find either a somewhat explicit expression for $\mathcal{C}_{\nu_0}(\beta)$ or a linear differential operator with polynomial coefficients annihilating $\mathcal{C}_{\nu_0}(\beta)$ which in a similar case the authors succeeded to obtain in [19]. (Observe that such an operator always exists since $\mathcal{C}_{\nu_0}(\beta)$ is a Nilsson-class function, see [16].)

3. CASE WHEN $\lim_{n \rightarrow \infty} \frac{a_n}{n^{2/3}} = a \neq 0$. PHYSICS RIGOR.

Fix $a \in \mathbb{C}$ and, similarly to the previous section, introduce a double indexed sequence $\{\Delta_n^{(k)}(a, \lambda)\}$ of the characteristic polynomials of the principal minors of $M_n^{(an^{2/3})}$, see (1.3). For any fixed a , we can easily obtain the corresponding recurrence relation of length 4:

$$\begin{aligned} \Delta_n^{(k)}(an^{2/3}, \lambda) &= -\lambda \Delta_n^{(k-1)}(an^{2/3}, \lambda) - (k-1)(n-k+2)an^{2/3} \Delta_n^{(k-2)}(an^{2/3}, \lambda) \\ &\quad + (n-k+2)(n-k+3)(k-1)(k-2) \Delta_n^{(k-3)}(an^{2/3}, \lambda). \end{aligned}$$

The same scaling by $n^{4/3}$ of the eigenvalues of $M_n^{(an^{2/3})}$ (which we conjecture results in compactly supported limiting root-counting measure ν_a) leads to

$$\begin{aligned} \tilde{\Delta}_n^{(k)}(an^{2/3}, \beta) &= -\beta \tilde{\Delta}_n^{(k-1)}(an^{2/3}, \beta) - \frac{(k-1)(n-k+2)a}{n^2} \tilde{\Delta}_n^{(k-2)}(an^{2/3}, \beta) \\ &\quad + \frac{(n-k+2)(n-k+3)(k-1)(k-2)}{n^4} \tilde{\Delta}_n^{(k-3)}(an^{2/3}, \beta), \end{aligned} \quad (3.1)$$

where $\lambda = \beta n^{4/3}$. Observe that (3) is the recurrence relation for the sequence of characteristic polynomials of the principal minors of $\tilde{M}_n^{(an^{2/3})}$ given by

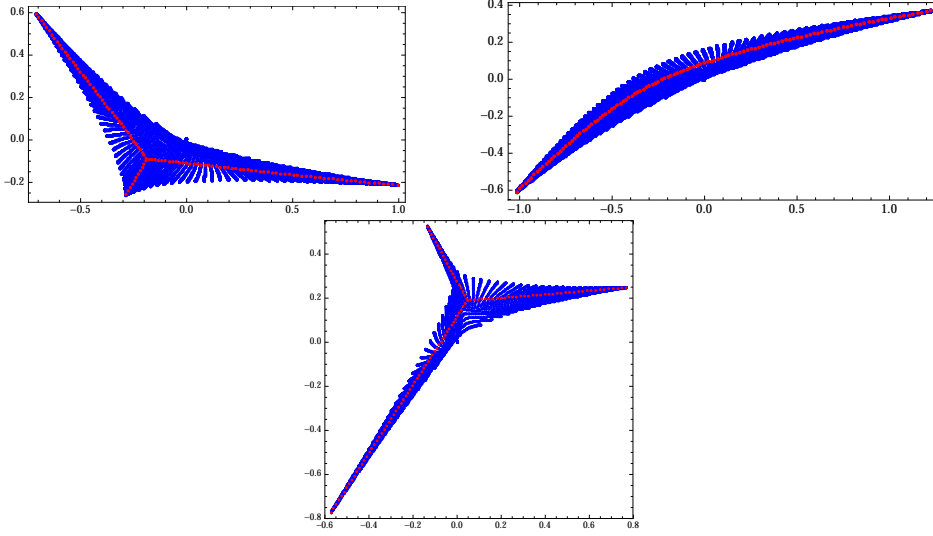


FIGURE 4. The (blue) union of the supports for (3.3) together with the (red) roots of $Sp_{200}(an^{2/3}, \beta n^{4/3})$ for $a = (1 - I)/2$, $a = I/2$ and $a = 1 + I$ resp.

$$\widetilde{M}_n^{(an^{2/3})} - \beta I := \begin{pmatrix} -\beta & \frac{a}{n^{2/3}} & \frac{2}{n^{4/3}} & 0 & 0 & \cdots & 0 \\ \frac{n}{n^{4/3}} & -\beta & \frac{2a}{n^{2/3}} & \frac{6}{n^{4/3}} & 0 & \cdots & 0 \\ 0 & \frac{n-1}{n^{4/3}} & -\beta & \frac{3a}{n^{2/3}} & \frac{12}{n^{4/3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{3}{n^{4/3}} & -\beta & \frac{(n-1)a}{n^{2/3}} & \frac{n(n-1)}{n^{4/3}} \\ 0 & 0 & \cdots & 0 & \frac{2}{n^{4/3}} & -\beta & \frac{na}{n^{2/3}} \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{n^{4/3}} & -\beta \end{pmatrix}. \quad (3.2)$$

Assuming that $\frac{k}{n} \rightarrow \tau$, we obtain that (3) tends to following relation with constant coefficients:

$$\widetilde{\Delta}_\tau^{(k)}(\beta) = -\beta \widetilde{\Delta}_\tau^{(k-1)}(\beta) - a\tau(1-\tau)\widetilde{\Delta}_\tau^{(k-2)} + (1-\tau)^2\tau^2\widetilde{\Delta}_\tau^{(k-3)}(\beta), \quad (3.3)$$

whose characteristic equation is given by

$$\Psi^3 + \beta\Psi^2 + a\tau(1-\tau)\Psi - (1-\tau)^2\tau^2 = 0. \quad (3.4)$$

Typically the union of the supports of the asymptotic root-counting measures for the polynomial sequences (3.3) depending on $\tau \in [0, 1]$ is larger than that of ν_a , see Fig. 4. In such case we can only conclude that the corresponding Cauchy transforms coincide in the complement to the larger support.

However, in case $a \geq \frac{3}{\sqrt[3]{4}}$ (which fits the situation covered by the main result of [12],) these supports coincide and one can obtain the density of ν_a by averaging the densities of (3.3).

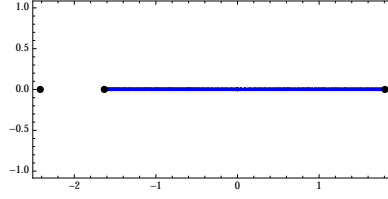


FIGURE 5. The union of the supports for (3.3) in case $a = 3$ together with 3 branching points given by (3.5).

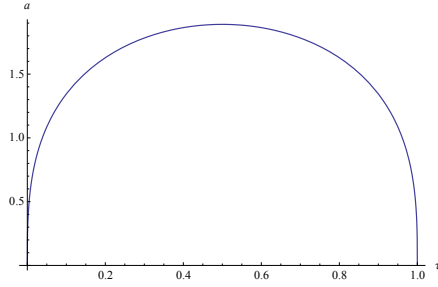


FIGURE 6. Plot of $a^3 - 27\tau + 27\tau^2 = 0$ in the real (a, τ) -plane.

Lemma 5. *If $a \geq \frac{3}{\sqrt[3]{4}}$, then for $\tau \in [0, 1]$, each support of the family (3.4) is a real interval. Their union coincides with the (real) interval connecting two rightmost branching points defined by (3.8), see Fig 5.*

Proof. Using symbolic manipulations, one can check that all three branching points of (3.4) with respect to Ψ satisfy the equation:

$$4\beta^3 + a^2\beta^2 - 18a\beta\tau(1 - \tau) + \tau(1 - \tau)(27\tau^2 - 27\tau - 4a^3) = 0. \quad (3.5)$$

To check that for $a \geq \frac{3}{\sqrt[3]{4}}$, and any $\tau \in [0, 1]$, all three solutions of (3.5) are real, we calculate the discriminant of (3.5) with respect to β . Again using symbolic manipulations, we get that this discriminant is given by:

$$Dsc := 16\tau(1 - \tau)(a^3 - 27\tau + 27\tau^2)^3.$$

For $\tau \in [0, 1]$, the graph of Dsc in the real (a, τ) -plane is presented in Fig. 6. One can easily check that the maximal value of a on this graph is obtained when $\tau = 1/2$ and is equal to $\frac{3}{\sqrt[3]{4}}$. At the same time, for $a = 3$ one can easily check that all three roots of (3.5) are real. \square

Remark 4. Similarly to the case $a = 0$, for any $a \geq \frac{3}{\sqrt[3]{4}}$, one can represent the Cauchy transform of ν_a in the complement to its support (which is an interval explicitly given in Lemma 5) as

$$\mathcal{C}_{\nu_0}(\beta) = \int_0^1 \frac{\tilde{\Psi}'_{\beta} d\tau}{\tilde{\Psi}},$$

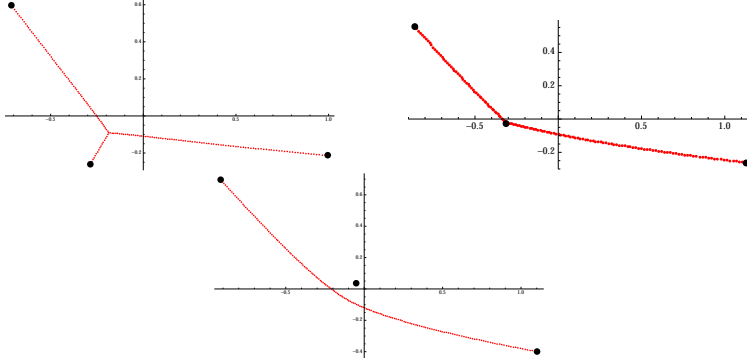


FIGURE 7. Root distributions of $Sp_{200}(an^{2/3}, \beta n^{4/3})$ for $a = (1 - I)/2$ (left), $a = 4/5 - 2I/3$ (right) and $a = 2/3 - I$ (down) in the β -plane. Larger dots are the endpoints of the support given by (3.8).

where $\tilde{\Psi}$ is the unique solution of (3.4) satisfying the condition

$$\lim_{\beta \rightarrow \infty} \frac{\tilde{\Psi}}{\beta} = -1.$$

(In an appropriate domain in \mathbb{C} such presentation for the Cauchy transform of ν_a is valid for any complex a .)

”Physics proof” of Proposition 2. To obtain the support of ν_a we argue as follows. Assume that we have a (sub)sequence $\beta_{j_n, n}$ of the eigenvalues of $\widetilde{M}_n(an^{2/3})$ (one for each n) converging to some finite limit denoted by Λ . Denote by $\{p_n\}$ the corresponding (sub)sequence of eigenpolynomials. (These eigenpolynomials are classically referred to as Stieltjes polynomials and in special cases their properties are well studied in the literature.) Each p_n satisfies its own differential equation:

$$p_n'' - (x^2 - an^{2/3})p_n' + (nx - \beta_{j_n, n}n^{4/3})p_n = 0.$$

First assumption. We assume that if the subsequence $\{\beta_{j_n, n}\}$ has a finite limit then the sequence $\{\mu_n\}$ of the root-counting measures of $\{p_n\}$ also weakly converges, after appropriate scaling of x , to some limiting measure $\kappa_{a, \Lambda}$ whose support consists of finitely many compact curves and points.

This assumption implies that the sequence of Cauchy transforms of scaled $\{\mu_n\}$ converges to the Cauchy transform of the limiting measure $\kappa_{a, \Lambda}$. The appropriate scaling of x can be easily guessed from the latter equation. Namely, substituting $x = \Theta n^{1/3}$ and dividing the above equation by $n^{4/3}p_n$, we get the following relation:

$$\frac{d^2 p_n}{d^2 \Theta} - (\Theta^2 - a) \frac{dp_n}{d\Theta} + (\Theta - \beta_{j_n, n}) = 0 \quad (3.6)$$

with respect to the new independent variable Θ . Observe that the scaled logarithmic derivative $\frac{dp_n}{np_n}$ is the Cauchy transform of the root-counting measure of $p_n(\Theta n^{1/3})$ with respect to the variable Θ .

Second assumption. Assuming that the sequence $\{\mu_n\}$ of the root-counting measures of $\{p_n(\Theta n^{1/3})\}$ converges to $\kappa_{a,\Lambda}$, we additionally assume that the sequences of the root-counting measures of its first and second derivatives converge to the same measure $\kappa_{a,\Lambda}$.

(Apparently this assumption can be rigorously proved by using the same arguments as presented in [3].)

Under the above assumptions, using (3.6), we get that the Cauchy transform $\mathcal{C}_{a,\Lambda}$ of $\kappa_{a,\Lambda}$ satisfies the quadratic equation:

$$\mathcal{C}_{a,\Lambda}^2 - (\Theta^2 - a)\mathcal{C}_{a,\Lambda} + (\Theta - \Lambda) = 0 \quad (3.7)$$

almost everywhere in \mathbb{C} . Up to a variable change $x \leftrightarrow \Theta$ and $\Lambda \leftrightarrow \beta$ the latter equation coincides with (1.4).

Third assumption. So far we presented a physics argument showing that if a (sub)sequence $\{\beta_{j_n,n}\}$ of the eigenvalues of $\widehat{M}_n^{(an^{2/3})}$ converges to some limit Λ , then there exists a probability measure $\kappa_{a,\Lambda}$ whose Cauchy transform satisfies (3.7) almost everywhere in the Θ -plane. Our final assumption is that the converse to the latter statement is true as well, i.e., for each Λ with the above properties there exists an appropriate subsequence $\{\beta_{j_n,n}\}$ converging to Λ . (At the moment we do not have a mathematically rigorous proof of this statement but such fact in a very similar situation was settled in [22].) \square

3.1. Quadratic equations with polynomial coefficients and quadratic differentials. The next result is a special case of Proposition 9 and Theorem 12 in [5].

Proposition 6. *There exists a signed measure $\mu_{a,\Lambda}$ whose Cauchy transform satisfies (3.7) almost everywhere if and only if the set of critical horizontal trajectories of the quadratic differential*

$$-((\Theta^2 - a)^2 - 4(\Theta - \Lambda))d\Theta^2$$

contains all its turning points, i.e., all roots of $P(\Theta, \Lambda) = (\Theta^2 - a)^2 - 4(\Theta - \Lambda)$. (Here by a critical horizontal trajectory of a quadratic differential we mean its horizontal trajectory which starts and ends at the turning points.)

We know that the support of $\mu_{a,\Lambda}$ should include all the branching points of (3.7) and consists of the critical horizontal trajectories of (1.5).

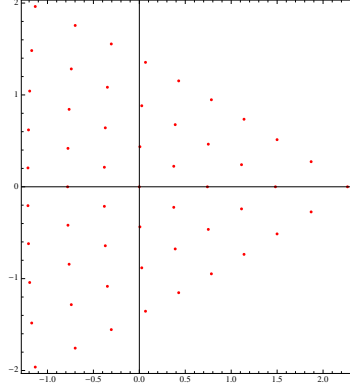


FIGURE 8. The triangle of the branching points for $Sp_{10}(a, \lambda)$.

Lemma 7. *The set of the critical values of the polynomial $P(\Theta, \Lambda) = ((\Theta^2 - a)^2 - 4(\Theta - \Lambda))$, i.e., the set of all Λ for which $P(\Theta, \Lambda)$ has a double root with respect to Θ is given by the equation:*

$$4\Lambda^3 + a^2\Lambda^2 - 9a\Lambda/2 - a^3 - 27/16 = 0. \quad (3.8)$$

Proof. Straight-forward calculations. \square

Corollary 1. *The endpoints of the support of ν_a are contained among the three roots of equation (3.5) when $\tau = 1/2$, see Fig. 7. This equation coincides with (3.8) where Λ is substituted by β .*

4. ON BRANCHING POINTS AND MONODROMY OF THE SPECTRUM

4.1. Monodromy of $Sp_n(a, \lambda)$. For any positive integer n and generic values of parameter a , the roots of $Sp_n(a, \lambda)$ with respect to λ (i.e., the quasi-exactly solvable spectrum) are simple. For any given n , and any sufficiently large positive a , these roots are real and distinct. The set $\Sigma_n \subset \mathbb{C}$ of branching points of $Sp_n(a, \lambda)$, i.e., the set of all values of a for which two eigenvalues coalesce, has cardinality $\binom{n+1}{2}$. When plotted these branching points form a regular pattern in the complex plane shown in Figures 2 and 8.

In this subsection we present our (mostly) numerical results and conjectures about the monodromy of the roots of $Sp_n(a, \lambda)$ when a runs along different closed paths in the complement to Σ_n in the a -plane. We start with the following statement.

Proposition 8. *For any given n , if $|a| \rightarrow \infty$ with $\arg a = \phi$ fixed, then the roots of $Sp_n(\lambda, a)$ divided by $n^{4/3}$ will be asymptotically uniformly distributed on the straight segment $[-\sqrt{a}, +\sqrt{a}]$, see Fig. 9. In particular, if a runs over the circle $Re^{2\pi it}$, $t \in [0, 1]$ for any sufficiently large R , then the resulting monodromy of roots of $Sp_n(\lambda, R)$ (which are all real) is the complete reversing of their order, i.e., the leftmost and the*

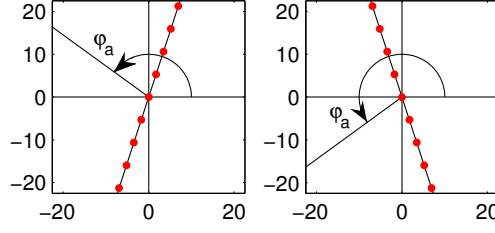


FIGURE 9. Roots of $Sp_8(a, 8^{4/3}\beta)$. The left figure shows the situation with $a = 500 \exp(i\varphi_a)$, $\varphi_a = 4\pi/5$, the right one $a = 500 \exp(i\varphi_a)$, $\varphi_a = 6\pi/5$. In both cases roots are almost uniformly distributed on the interval $[-\sqrt{a}, \sqrt{a}]$.

rightmost roots change places, the second from the left and the second from the right change places etc.

Proof of Proposition 8. Observe that if $a = Ke^{2\pi i\phi}$ with K very large then polynomial $Sp_n(\beta, a)$, comp. (2.15), is close to $\widehat{Sp}_n(\beta, a)$ where $\widehat{Sp}_n(\beta, a)$ is the characteristic polynomial of the tridiagonal matrix

$$\widehat{M}_n^{(a)} := \begin{pmatrix} 0 & a/n & 0 & 0 & 0 & \cdots & 0 \\ n/n & 0 & 2a/n & 0 & 0 & \cdots & 0 \\ 0 & (n-1)/n & 0 & 3a/n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 3/n & 0 & (n-1)a/n & 0 \\ 0 & 0 & \cdots & 0 & 2/n & 0 & na/n \\ 0 & 0 & \cdots & 0 & 0 & 1/n & 0 \end{pmatrix}. \quad (4.1)$$

To make the situation more transparent, let us consider the sequence of characteristic polynomials of $\frac{1}{\sqrt{a}}\widehat{M}_n^{(a)}$ and of $\frac{1}{\sqrt{a}}\widetilde{M}_n^{(a)}$. In other words, we are comparing the roots of $\widehat{Sp}_n(\beta, a)$ divided by \sqrt{a} with that of $\widetilde{Sp}_n(\beta, a)$ divided by \sqrt{a} . The characteristic polynomials of the respective principal minors of $\frac{1}{\sqrt{a}}\widehat{M}_n^{(a)}$ and of $\frac{1}{\sqrt{a}}\widetilde{M}_n^{(a)}$ satisfy the recurrences:

$$\begin{aligned} \widetilde{\Delta}_n^{(k)}(\gamma) &= -\gamma \widetilde{\Delta}_n^{(k-1)}(\gamma) - \frac{(k-1)(n-k+2)}{n^{8/3}} \widetilde{\Delta}_n^{(k-2)}(\gamma) \\ &\quad + \frac{(n-k+2)(n-k+3)(k-1)(k-2)}{a^{3/2}n^4} \widetilde{\Delta}_n^{(k-3)}(\gamma), \end{aligned} \quad (4.2)$$

and

$$\widehat{\Delta}_n^{(k)}(\gamma) = -\gamma \widehat{\Delta}_n^{(k-1)}(\gamma) - \frac{(k-1)(n-k+2)}{n^{8/3}} \widehat{\Delta}_n^{(k-2)}(\gamma) \quad (4.3)$$

where $\gamma = \beta/\sqrt{a}$ and both recurrences have the standard boundary conditions: $\widetilde{\Delta}_n^{(-1)}(\gamma) = \widehat{\Delta}_n^{(-1)}(\gamma) = 0$, $\widetilde{\Delta}_n^{(0)}(\gamma) = \widehat{\Delta}_n^{(0)}(\gamma) = 1$. As before $\widetilde{Sp}_n(\gamma, a) = \widetilde{\Delta}_n^{(n)}(\gamma)$ and $\widehat{Sp}_n(\gamma, a) = \widehat{\Delta}_n^{(n)}(\gamma)$. Observe now that, for any fixed n and any $\epsilon > 0$, one can choose $|a|$ so large that (due to

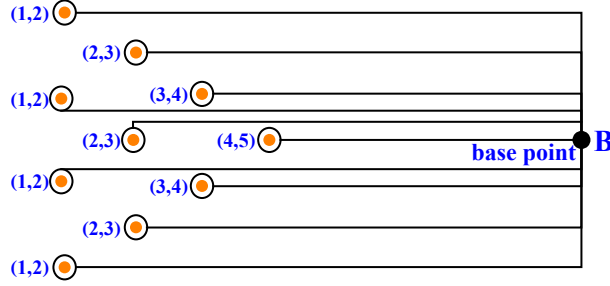


FIGURE 10. The system of standard paths and the monodromy (transpositions) of the spectrum which they produce, $n = 4$.

the presence of $a^{3/2}$ in the denominator of the third term in (4.1)) each equation in (4.1) for $k = 1, 2, \dots, n$ deviated from the corresponding equation in (4.3) so little that $\tilde{\Delta}_n^{(n)}(\gamma) - \hat{\Delta}_n^{(n)}(\gamma)$ can be made arbitrary small coefficientwise.

Now one can easily check by induction that bivariate polynomial $\widehat{Sp}_n(\lambda, a)$ is quasihomogeneous with weight 1 for variable λ and weight 2 for variable a . When a is positive, then $\widehat{Sp}_n(\lambda, a)$ is a real-rooted polynomial. Since multiplication of λ by $e^{\pi i t}$ and multiplication of a by $e^{2\pi i t}$ multiplies the whole $\widehat{Sp}_n(\lambda, a)$ by a constant, the roots of $\widehat{Sp}_n(\lambda, a)$ with respect to λ lie for any fixed a on the line through the origin whose slope is half of the slope of a . Consider the roots of $\widehat{Sp}_n(\beta, a)$ with respect to β . Observe that if $a = 1$, then the spectrum of (4.1) is $(-1, -1 + \frac{2}{n}, -1 + \frac{4}{n}, \dots, 1 - \frac{4}{n}, 1 - \frac{2}{n}, 1)$. The same argument as above gives that for any $a \neq 0$ the roots of $\widehat{Sp}_n(\beta, a)$ will be equally spaced on the interval $[-\sqrt{a}, \sqrt{a}]$.

Since choosing $|a|$ sufficiently large, we can achieve that all roots of $\tilde{Sp}_n^{(n)}(\gamma, a)$ lie arbitrarily close to that of $\widehat{Sp}_n^{(n)}(\gamma, a)$, and the latter roots are equally distributed on the interval $[e^{-2\pi i \phi}, e^{-2\pi i \phi}]$, the result follows. \square

To describe our conjecture on monodromy of the spectrum, let us introduce a system of standard paths connecting a base point which we choose as a sufficiently large positive number with every branching point, see Fig. 10. Based on our numerical experiments, we see that Σ_n has a triangular shape \mathfrak{F}_n with points regularly arranged into columns and rows in \mathbb{C} . There are n columns (enumerated from left to right) where the j -th column consists of $n - j$ branching points with approximately the same real part and there are n rows (enumerated from bottom to top) where the i -th row consists of points with approximately the same imaginary part. We denote the branching points

$\sigma_{i,j} \in \Sigma_n$ where $i = 1, \dots, 2n - 1$ is the row number and $j = 1, \dots, n$ is the column number.

Fixing a base point B , connect B with every $\sigma_{i,j}$ by a "vertical hook" $\mathcal{P}_{i,j}$, i.e., move from B vertically to the imaginary part of $\sigma_{i,j}$, then move horizontally to the left till you almost hit $\sigma_{i,j}$, then circumgo $\sigma_{i,j}$ counterclockwise along a small circle centered at $\sigma_{i,j}$ and return back to B along the same path. Conjecturally, along such a path one will never hit any other branching points unless $\sigma_{i,j}$ lies on the real axis. In other words, the imaginary parts of all branching points except for the real ones are all distinct. In case when $\sigma_{i,j}$ is real one can slightly deform the suggested path (which is a real interval) in an arbitrary way to move it away from the real axis. The resulting monodromy will (conjecturally) be independent of any such small deformation, see below.

Conjecture 2. *For any $\sigma_{i,j} \in \Sigma_n$ and any sufficiently large positive base point B , the monodromy corresponding to the standard path $\mathcal{P}_{i,j}$ is a simple transposition $(j, j + 1)$ of the roots of $Sp_n(\lambda, B)$ (which are all real) ordered from left to right.*

This conjecture has been numerically checked by the second author for all $n \leq 10$. Observe that since the system of standard paths gives a basis of the fundamental group $\pi_1(\mathbb{C} \setminus \Sigma_n)$, then knowing the monodromy for the standard paths, one can calculate the monodromy along any loop in $\mathbb{C} \setminus \Sigma_n$ based at B .

4.2. Asymptotic distribution of branching points of $Sp_n(a, \lambda)$ and Yablonskii-Vorob'ev polynomials. There are two natural asymptotic regimes for the sequence $\{\Sigma_n\}$ when $n \rightarrow \infty$. In the first regime we just consider the sequence $\{\Sigma_n\}$ and study the stabilization of its portion in any fixed bounded domain of \mathbb{C} when $n \rightarrow \infty$. In the second regime we consider the sequence $\{\Sigma_n/n^{2/3}\}$ and study the limiting density of zeros in the triangular shape \mathfrak{F} which is more and more densely filled by the roots of $\Sigma_n/n^{2/3}$ when $n \rightarrow \infty$. Numerical experiments show that both limits exist.

Conjecture 3. The sequence $\{\Sigma_n\}$ splits into three subsequences depending on $n \bmod 3$ and each subsequence stabilizes to its own hexagonal lattice in \mathbb{C} , see Fig. 11.

Conjecture 4. The sequences $\{\Sigma_n/(\frac{27}{4})^{1/3}n^{2/3}\}$ and $\{\mathcal{Z}_n/(\frac{9}{2})^{2/3}n^{2/3}\}$ when $n \rightarrow \infty$ asymptotically fills in the curvilinear triangular shape \mathfrak{F} , see Fig 2. The interior of \mathfrak{F} is the set of all values $a \in \mathbb{C}$ for which the support of ν_a consists of three smooth segments=legs with a common point, see Fig. 7 (left). The complement of \mathfrak{F} consists of all values $a \in \mathbb{C}$ for which the support of ν_a is a single smooth segment, see Fig. 7 (down). The boundary of \mathfrak{F} consists of those $a \in \mathbb{C}$ for which

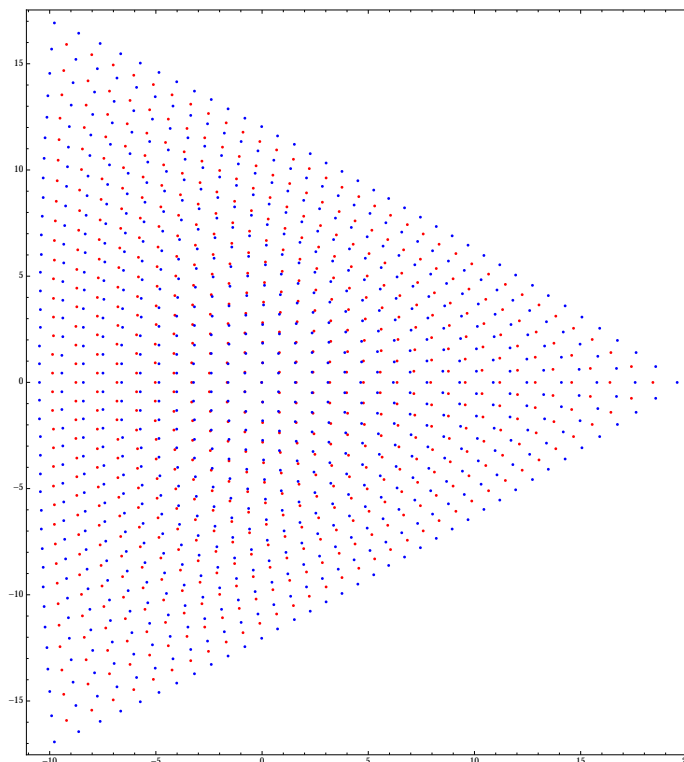


FIGURE 11. Stabilization of the lattice, $n = 34$ and $n = 37$.

the support of ν_a is a single curve with a singularity, i.e. one of the three legs disappears, see Fig. 7 (right).

The above statement clarifies and strengthens Conjecture 1 from the Introduction.

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